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On the regularity of solutions to integral equations with nonsmooth kernels on a union of open intervals ☆

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ABSTRACT

The behaviour of a solution to a Fredholm integral equation of the second kind on a union of open intervals is examined. The kernel of the corresponding integral operator may have diagonal singularities, information about them is given through certain estimates. The weighted spaces of smooth functions with boundary singularities containing the solution of the integral equation are described.

1. Introduction

In this work we provide an analysis of Fredholm integral equations of the form

$$u(x) = \int_{\mathcal{G}} K(x,y)u(y)\mathrm{d}y + f(x), \quad x \in \mathcal{G}, \tag{1}$$

where $\mathcal{G}$ is a finite union of open intervals,

$$\mathcal{G} = \bigcup_{i=1}^{n} (a_i, b_i), \quad a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_{n-1} < b_{n-1} \le a_n < b_n. \tag{2}$$

In particular, the linear transport equation in slab geometry, in the case of a multilayered system composed of a finite number of different homogeneous slabs, is of the type of problem (1) and (2), see [2,7,9].

In the case $n = 1$, Eq. (1) is a standard linear Fredholm integral equation of the second kind on a bounded interval (a_1, b_1).

The purpose of the present paper is to study the regularity properties of a function $u(x)$ defined on $\mathcal{G}$ as a solution to (1) in the case where the kernel $K(x, y)$ is at most weakly singular at $x = y$, see Section 2. Assuming certain differentiability properties of the kernel K and forcing function f, we estimate the growth rates of the derivatives of $u(x)$ as x approaches the boundary $\partial\mathcal{G} = \{a_1, b_1, \ldots, a_n, b_n\}$ of $\mathcal{G}$, see Theorems 5, 10 and 11. These results have particularly important applications for the problem of solving (1) numerically.

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For different special cases it is well documented how a diagonal singularity of the kernel of an integral equation of the second kind generates boundary singularities of the solution (more precisely, of the derivatives of the solution). The case of one-dimensional Fredholm integral equations has been analysed in [1,8,10–13,18,20,22–24,27,28], the case of Volterra integral equations in [3–6,16] and the case of multidimensional integral equations in [14,19,21,25,26].

Note that Theorem 5 below is most closely related to the corresponding results of the work [26]. In fact, the class of kernels $W^{m,\nu}$ allowed in the present paper is an adaption of the class of functions that were introduced in [26] for N-dimensional integral equations with weakly singular kernels. However, the analysis of [26] essentially exploits the assumption $N \geq 2$. Unexpectedly, in the case $N = 1$, some details of the analysis turn out to be more sophisticated than in case $N \geq 2$. The approach elaborated in [26] for an integral equation of the form (1) is based on the smallness of $\int_\Omega K(x, y)u(y)\mathrm{d}y$, $x \in \Omega$, where $\Omega \subset \mathcal{G} \subset \mathbb{R}^N$ ($N \geq 2$) is a small subregion; the integral over $\mathcal{G} \setminus \Omega$ is treated as an additional part of the forcing function f. Our approach below is different from that in [26]: we introduce the appropriate weighted spaces of smooth functions and show that the integral operator is compact in these spaces. Thus, since Theorem 5 has not been proven in [26], we present a detailed proof of Theorem 4 which is the key to the proof of Theorem 5, see Sections 4–6.

We present also some results characterizing a new phenomenon of the behaviour of the derivatives of $u(x)$ near the "inner boundary" of $\mathcal{G}$ in case of further assumptions on the kernel K, see Theorem 10. Again, the key to the proof of Theorem 10 is the compactness of the integral operator in the appropriate weighted spaces, see Theorem 9.

Finally, we extend our analysis to an eigenvalue problem for Eq. (1) if $f = 0$. The results obtained are given in Theorem 11.

Throughout the paper we denote $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{N} = \{1, 2, \ldots\}$. By c, c', c_1 etc. we denote generic constants that may have different values by different occurrences; we write c_K if we want to point out that the constant c may depend on the kernel K, etc.

2. Class of kernels

Let

$$\mathrm{diag} = \mathrm{diag}\,\mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 : x = y\}$$

be the diagonal of $\mathbb{R}^2$. We are interested in kernels $K(x, y)$ of Eq. (1) that are smooth outside diag and may have a weak singularity at $x = y$. Actually, we assume that $K \in W^{m,\nu}$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $-\infty < \nu < 1$.

Here, $\mathcal{W}^{m,\nu} = \mathcal{W}^{m,\nu}((\mathcal{G} \times \mathcal{G}) \setminus \mathrm{diag})$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$, is defined as the collection of m times continuously differentiable functions K on $(\mathcal{G} \times \mathcal{G}) \setminus \mathrm{diag}$ that satisfy there for all $k, l \in \{0\} \cup \mathbb{N}$, $k + l \leq m$, the inequalities

$$\left|\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l K(x, y)\right| \leq c_{m,K} \begin{cases} 1 & \text{if } \nu + k < 0, \\ 1 + |\log|x - y|| & \text{if } \nu + k = 0, \\ |x - y|^{-\nu - k} & \text{if } \nu + k > 0. \end{cases} \tag{3}$$

A consequence of (3) is that

$$\left|\left(\frac{\partial}{\partial y}\right)^k \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l K(x, y)\right| \leq c'_{m,K} \begin{cases} 1 & \text{if } \nu + k < 0, \\ 1 + |\log|x - y|| & \text{if } \nu + k = 0, \\ |x - y|^{-\nu - k} & \text{if } \nu + k > 0. \end{cases} \tag{4}$$

For $k = l = 0$, condition (3) yields

$$|K(x, y)| \leq c_{m,K} \begin{Bmatrix} 1, & \nu < 0 \\ 1 + |\log|x - y||, & \nu = 0 \\ |x - y|^{-\nu}, & \nu > 0 \end{Bmatrix}, \quad x, y \in \mathcal{G},\ x \neq y. \tag{5}$$

It follows from (5) that a kernel $K \in \mathcal{W}^{m,\nu}$ is at most weakly singular for $0 \leq \nu < 1$. For $\nu < 0$, the kernel $K \in W^{m,\nu}$ is bounded on $(\mathcal{G} \times \mathcal{G})\setminus\mathrm{diag}$ but its derivatives may have diagonal singularities. The most important examples of weakly singular kernels $K \in \mathcal{W}^{m,\nu}$ are given by

$$K(x, y) = g(x, y)|x - y|^{-\nu} \quad \text{for } 0 < \nu < 1,$$
$$K(x, y) = g(x, y) \log|x - y| \quad \text{for } \nu = 0,$$

where g is an m times continuously differentiable function on $(a_i, b_i) \times (a_j, b_j)$ $(i, j = 1, \ldots, n)$ such that $g(x, y)$ itself and its derivatives are bounded on $(a_i, b_i) \times (a_j, b_j)$ and may have different one-sided limits as x or y tends to a point of the form $a_i = b_{i-1}$ if $\mathcal{G}$ has an "inner boundary" $a_i = b_{i-1}$ for some i, $2 \leq i \leq n$.

Remark 1. Although we assume in the definition of $\mathcal{W}^{m,\nu}$ that K is given only for $(x, y) \in (\mathcal{G} \times \mathcal{G})\setminus\mathrm{diag}$, actually $K|_{((a_i,b_i)\times(a_j,b_j))\setminus\mathrm{diag}}$, the restriction of $K \in \mathcal{W}^{m,\nu}$ to the set $((a_i, b_i) \times (a_j, b_j)) \setminus \mathrm{diag}$, has a continuous extension to $([a_i, b_i] \times [a_j, b_j]) \setminus \mathrm{diag}$, $i, j = 1, \ldots, n$. (Different one-sided limits are possible if $a_i = b_{i-1}$ for some $2 \leq i \leq n$). Later such extensions of K will be denoted again by K.

3. Weighted spaces of smooth functions

Let $m \in \mathbb{N}$, $-\infty < \nu < 1$, $-\infty < a < b < \infty$. By $C^{m,\nu}(a,b)$ we denote the set of m times continuously differentiable functions u on (a,b) that satisfy the inequalities

$$\left|u^{(k)}(x)\right| \le c_u \begin{Bmatrix} 1 & \text{if } k < 1-\nu \\ 1 + |\log \rho_{a,b}(x)| & \text{if } k = 1-\nu \\ \rho_{a,b}(x)^{1-\nu-k} & \text{if } k > 1-\nu \end{Bmatrix}, \quad k = 0, 1, \ldots, m. \tag{6}$$

Here $a < x < b$ and

$$\rho_{a,b}(x) = \min\{x-a, b-x\}$$

is the distance from $x \in (a,b)$ to the boundary of the interval (a,b). For $s \in \mathbb{R}$, define the following weight functions on (a,b):

$$w_s^{(a,b)}(x) = \begin{Bmatrix} 1 & \text{if } s < 0 \\ \dfrac{1}{1 + |\log \rho_{a,b}(x)|} & \text{if } s = 0 \\ \rho_{a,b}(x)^s & \text{if } s > 0 \end{Bmatrix}, \quad a < x < b. \tag{7}$$

Equipped with the norm

$$\|u\|_{C^{m,\nu}(a,b)} = \sum_{k=0}^{m} \sup_{a<x<b} w_{k+\nu-1}^{(a,b)}(x)|u^{(k)}(x)|, \quad u \in C^{m,\nu}(a,b), \tag{8}$$

$C^{m,\nu}(a,b)$ becomes a Banach space. We will also use a simplified norm in $C^{m,\nu}(a,b)$ which is equivalent to the norm (8).

Lemma 2 ([20]). *Let $a < x_1 < \cdots < x_m < b$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$. Then the basic norm (8) is equivalent to the norm*

$$\|u\|'_{C^{m,\nu}(a,b)} = \max_{j=1,\ldots,m} |u(x_j)| + \sup_{a<x<b} w_{m+\nu-1}^{(a,b)}(x)|u^{(m)}(x)|. \tag{9}$$

Moreover, the following conditions (i) *and* (ii) *are equivalent for a set $\mathcal{M} \subset C^{m,\nu}(a,b)$:*

(i) *$\mathcal{M}$ is relatively compact in $C^{m,\nu}(a,b)$;*
(ii) *the functions v from $\mathcal{M}$ are m times continuously differentiable in (a,b), uniformly bounded at $x_1, \ldots, x_m$ and the set $\{w_{m+\nu-1}^{(a,b)} v^{(m)} : v \in \mathcal{M}\}$ is relatively compact in $BC(a,b)$.*

Note that $BC(a,b)$ is the Banach space of *b*ounded *c*ontinuous functions u on the open interval (a,b) equipped with the norm $\|u\|_{BC(a,b)} = \sup_{a<x<b} |u(x)|$.

Introduce also the following standard spaces of continuous functions:

$C^m(\Omega)$ $(m \ge 0)$ is the space of m times continuously differentiable functions on $\Omega \subset \mathbb{R}^N$, $C^0(\Omega) = C(\Omega)$;

$C[a,b]$ is the Banach space of *c*ontinuous functions u on the closed interval $[a,b]$ equipped with the norm $\|u\|_{C[a,b]} = \max_{a\le x\le b} |u(x)|$;

$UC(a,b)$ is the closed subspace of $BC(a,b)$ that consists of *u*niformly *c*ontinuous functions u on (a,b) equipped with the norm $\|u\|_{UC(a,b)} = \sup_{a<x<b} |u(x)|$.

Clearly, a continuous function u on (a,b) has a continuous extension to $[a,b]$ if and only if u is uniformly continuous on (a,b). This enables the identification of the spaces $UC(a,b)$ and $C[a,b]$. Note that

$$C^{m,\nu}(a,b) \subset C[a,b], \quad m \in \mathbb{N},\ \nu \in \mathbb{R},\ \nu < 1 \tag{10}$$

(where we identify $C[a,b]$ with $UC(a,b)$). Moreover, with the help of the Arzela Theorem we obtain that imbedding (10) (i.e. the corresponding imbedding operator) is compact.

Denote by $UC(\mathcal{G})$ the Banach space of functions on $\mathcal{G}$ that are uniformly continuous on each interval (a_i, b_i), $i = 1, \ldots, n$, equipped with the norm

$$\|u\|_{UC(\mathcal{G})} = \max_{i=1,\ldots,n} \sup_{a_i<x<b_i} |u(x)|, \quad u \in UC(\mathcal{G}).$$

A function $u \in UC(\mathcal{G})$ has finite one-sided limits at all boundary points a_i, b_i, $i = 1, \ldots, n$ (if $a_i = b_{i-1}$ for some $i \in \{2, \ldots, n\}$, these one-sided limits of u may be different at $a_i = b_{i-1}$).

Lemma 3. *If $K \in \mathcal{W}^{m,\nu}((\mathcal{G} \times \mathcal{G}) \setminus \mathrm{diag})$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$, then the integral operator T_K defined by*

$$(T_K u)(x) = \int_{\mathcal{G}} K(x,y)u(y)\mathrm{d}y, \quad x \in \mathcal{G} \tag{11}$$

is compact as an operator from $UC(\mathcal{G})$ into $UC(\mathcal{G})$.

The proof of Lemma 3 is standard, cf. [15].

For $m \in \mathbb{N}$ and $-\infty < \nu < 1$, we define the underlying Banach space $C^{m,\nu}(\mathcal{G})$ as follows: $C^{m,\nu}(\mathcal{G})$ consists of functions u on $\mathcal{G}$ such that

$$u|_{(a_i,b_i)} \in C^{m,\nu}(a_i, b_i), \quad i = 1, \dots, n,$$
$$\|u\|_{C^{m,\nu}(\mathcal{G})} = \max_{i=1,\dots,n} \|u|_{(a_i,b_i)}\|_{C^{m,\nu}(a_i,b_i)}. \tag{12}$$

Thus, the boundary singularities of the derivatives of a function $u \in C^{m,\nu}(\mathcal{G})$ ($m \in \mathbb{N}$, $\nu < 1$) are characterized by the inequalities

$$|u^{(k)}(x)| \le \|u\|_{C^{m,\nu}(\mathcal{G})} \begin{cases} 1 & \text{if } k < 1-\nu, \\ 1 + |\log(x-a_i)| + |\log(b_i - x)| & \text{if } k = 1-\nu, \\ (x-a_i)^{1-\nu-k} + (b_i - x)^{1-\nu-k} & \text{if} k > 1-\nu, \end{cases}$$

where $a_i < x < b_i$, $i = 1, \dots, n$ and $k = 0, 1, \dots, m$.

Below we need a special subspace of $C^{m,\nu}(\mathcal{G})$ in the case where $\mathcal{G}$ (see (2)) has an “inner boundary” $a_i = b_{i-1}$ for some $i \in \{2, \dots, n\}$. Here for simplicity of the presentation we will assume that $n = 2$ and denote $a_1 = a$, $a_2 = b_1 = d$, $b_2 = b$, i.e. $\mathcal{G} = \mathcal{G}_d$ where

$$\mathcal{G}_d \equiv (a, d) \cup (d, b) = (a, b) \setminus \{d\}, \quad a < d < b.$$

Let $e \in C[a, b]$ be a cutting function such that $0 \le e(x) \le 1$ for $a \le x \le b$, $e(x) = 1$ in the vicinity of a and b, and $e(x) = 0$ in the vicinity of d, e.g.

$$e(x) = 1 \quad \text{for } x \in \left[a, a + \frac{d-a}{4}\right] \cup \left[b - \frac{b-d}{4}, b\right],$$
$$e(x) = 0 \quad \text{for } x \in \left[\frac{a+d}{2}, \frac{d+b}{2}\right].$$

In order to characterize growth rates of the derivatives of a function $u(x)$ as $x \in G_d$ approaches the point d we introduce, in addition to (7), the weight functions

$$w_s^{(d)}(x) = \left\{ \begin{array}{ll} 1 & \text{if } s < 0 \\ \dfrac{1}{1 + |\log \rho_d(x)|} & \text{if } s = 0 \\ \rho_d(x)^s & \text{if } s > 0 \end{array} \right\}, \quad x \in \mathcal{G}_d, s \in \mathbb{R},$$

where $\rho_d(x) = |x - d|$, $x \in \mathcal{G}_d$.

For $m, p \in \mathbb{N}$, $p \le m$, $\nu \in \mathbb{R}$, $\nu < 1$, denote by $C^{m,\nu,p}(\mathcal{G}_d)$ the Banach space of functions $u \in C^m(\mathcal{G}_d) \cap C^p(a, b)$ such that

$$\|u\|_{m,\nu,p} \equiv \sum_{j=0}^{m} \sup_{x \in \mathcal{G}_d} e(x)\, w_{j+\nu-1}^{(a,b)}(x)|u^{(j)}(x)| + \sum_{j=0}^{m} \sup_{x \in \mathcal{G}_d} (1 - e(x))\, w_{j+\nu-1-p}^{(d)}(x)|u^{(j)}(x)| < \infty.$$

Clearly,

$$C^{m,\nu}(a, b) \subset C^{m,\nu,p}(\mathcal{G}_d) \subset C^{m,\nu}(\mathcal{G}_d), \quad m, p \in \mathbb{N},\ p \le m,\ \nu < 1, \tag{13}$$

and these imbeddings are bounded.

4. Main results

The main results of the present paper are formulated in Theorems 4, 5 and 9–11.

Theorem 4. *Let $K \in \mathcal{W}^{m,\nu}((\mathcal{G} \times \mathcal{G}) \setminus \text{diag})$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$. Let T_K be defined by the formula (11). Then $T_K : C^{m,\nu}(\mathcal{G}) \to C^{m,\nu}(\mathcal{G})$ is compact.*

The proof of Theorem 4 is given in Section 6. In Section 5 we present some auxiliary results needed for the proof of Theorem 4.

Since we are interested also in the eigenvalue problem for T_K, we do not assume the uniqueness of the solution in the following theorem.

Theorem 5. *Let $K \in \mathcal{W}^{m,\nu}((\mathcal{G} \times \mathcal{G}) \setminus \text{diag})$ and $f \in C^{m,\nu}(\mathcal{G})$ where $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$. Then any solution $u \in UC(\mathcal{G})$ of equation (1) belongs to $C^{m,\nu}(\mathcal{G})$.*

For the proof of Theorem 5 we will use the following result.

Lemma 6. *Let E and F be Banach spaces such that $E \subset F$ densely and continuously, i.e., E is dense in F and $\|u\|_F \le c\|u\|_E$ for every $u \in E$. Let T be a linear operator in F that maps E into E and, moreover, let $T : E \to E$ and $T : F \to F$ be compact. Assume that the equation $u = Tu + f$ with a given $f \in E$ has a solution $u \in F$. Then $u \in E$.*

This Lemma follows from the Fredholm theory for compact operators, see [28] or Section 8 for a detailed proof. The claim of Lemma 6 is clear in the case where the homogeneous equation $u = Tu$ has only the trivial solution $u = 0$ in F. But we avoid this assumption in order to have a possibility to tackle the smoothness properties of eigenfunctions of the integral operator T_K, see Theorem 11 below.

Proof of Theorem 5. Denote $F = UC(\mathcal{G})$, $E = C^{m,\nu}(\mathcal{G})$ and $T = T_K$. Since the imbedding $C^{m,\nu}(a_i, b_i) \subset C[a_i, b_i]$ $(i = 1, \dots, n)$ is compact, the imbedding $E \subset F$ is continuous, even compact; since $C^m[a_i, b_i] \subset C^{m,\nu}(a_i, b_i)$, $i = 1, \dots, n$, $E \subset F$ densely. Now the statement of Theorem 5 follows from Lemmas 6 and 3 and Theorem 4. □

Remark 7. If $n = 1$ and $0 \le \nu < 1$, then the statement of Theorem 5 follows also from Theorems 1.2 and 1.3 of [20].

Remark 8. In general, the claim of Theorem 5 cannot be strengthened: assuming $f|_{[a_i,b_i]} \in C^m[a_i, b_i]$, $i = 1, \dots, n$ (or even $f|_{[a_i,b_i]} \in C^\infty[a_i, b_i]$, $i = 1, \dots, n$), a solution to (1) in general still does have the characteristic singularities of functions from $C^{m,\nu}(\mathcal{G})$.

Theorem 9. *Let $\mathcal{G} = \mathcal{G}_d = (a, b) \setminus \{d\}$, $a < d < b$, and let*

$$K \in \mathcal{W}^{m,\nu}((\mathcal{G}_d \times \mathcal{G}_d) \setminus \mathrm{diag}) \cap C^{p-1}(((a, b) \times (a, b)) \setminus \mathrm{diag}),$$

where $m, p \in \mathbb{N}$, $p \le m$, $\nu \in \mathbb{R}$, $\nu < 1$. Then T_K defined by (11) is compact as an operator from $C^{m,\nu,p}(\mathcal{G}_d)$ into $C^{m,\nu,p}(\mathcal{G}_d)$.

The proof of Theorem 9 is given in Section 7.

Theorem 10. *Let the assumptions of Theorem 9 be fulfilled. Let $f \in C^{m,\nu,p}(\mathcal{G}_d)$. Then any solution $u \in UC(\mathcal{G}_d)$ of equation (1) belongs to $C^{m,\nu,p}(\mathcal{G}_d)$.*

Proof. Denote $F = UC(\mathcal{G}_d)$, $E = C^{m,p,\nu}(\mathcal{G}_d)$ and $T = T_K$. Now the statement of Theorem 10 follows from Lemmas 6 and 3 and Theorem 9. □

Let us direct attention to a corollary of Theorems 5 and 10. Often the kernel $K(x, y)$ has a weak diagonal singularity for $x, y \in [a, b]$, whereas the forcing term f has a singularity at some points in (a, b), see, e.g. [17]. For example, let $f(x)$ or/and its derivatives be singular at $x = d$, $a < d < b$. We still can apply Theorems 5 and 10 obtaining that $f \in C^{m,\nu}(\mathcal{G}_d)$ or $f \in C^{m,\nu,p}(\mathcal{G}_d)$ implies respectively that $u \in C^{m,\nu}(\mathcal{G}_d)$ or $u \in C^{m,\nu,p}(\mathcal{G}_d)$ for a solution $u \in UC(\mathcal{G}_d)$ of Eq. (1) where $\mathcal{G} = \mathcal{G}_d$, $\mathcal{G}_d = (a, b) \setminus \{d\}$.

Note also that we have not assumed the uniqueness of the solution u of Eq. (1) in Theorems 5 and 10. With $f = 0$, these theorems can be applied to characterize the singularities of eigenfunctions of the operator T_K defined by (11). Under the conditions of Theorems 5 and 10 the operator T_K is linear and compact as an operator from $UC(\mathcal{G})$ into $UC(\mathcal{G})$. Therefore each point $z_0 \ne 0$ of the spectrum of T_K is an isolated eigenvalue of T_K and the generalized eigenspace $V(z_0, T_K)$ of T_K, corresponding to the eigenvalue $z_0 \ne 0$,

$$V(z_0, T_K) = \bigcup_{j=1}^{\infty} N((z_0 I - T_K)^j) \subset UC(\mathcal{G}), \tag{14}$$

is finite dimensional (and the union in (14) is actually finite). Here I is the identity mapping and $N(z_0 I - T_K) = \{u \in UC(\mathcal{G}) : (z_0 I - T_K)u = 0\}$. By induction over j in the union (14), Theorems 5 and 10 imply the following result.

Theorem 11. *Let $z_0 \ne 0$ be an eigenvalue of the operator T_K defined by (11). Then the following is true: under the conditions of Theorem 4, $V(z_0, T_K) \subset C^{m,\nu}(\mathcal{G})$; under the conditions of Theorem 9, $V(z_0, T_K) \subset C^{m,\nu,p}(\mathcal{G}_d)$.*

5. Differentiation of weakly singular integrals

In this section we derive some auxiliary results needed for the proof of Theorem 4. First we recall a differentiation result for weakly singular integrals with respect to a parameter.

Lemma 12 ([20]). *Assume that $g(x, y)$ is a continuously differentiable function on $((a, b) \times [a, b]) \setminus \mathrm{diag}$ and satisfies there the inequalities*

$$|g(x, y)| \le c|x - y|^{-\nu}, \qquad \left|\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) g(x, y)\right| \le c|x - y|^{-\nu}, \tag{15}$$

with some constants $c > 0$ and $0 < \nu < 1$. Then the function $x \to \int_a^b g(x, y)\mathrm{d}y$ is continuously differentiable in (a, b) and

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^b g(x, y)\mathrm{d}y = \int_a^b \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) g(x, y)\mathrm{d}y + g(x, a) - g(x, b), \quad a < x < b. \tag{16}$$

Let T_K be defined by the formula (11) where

$$K \in \mathcal{W}^{m,\nu}((\mathcal{G} \times \mathcal{G}) \setminus \mathrm{diag}), \quad u \in C^{m,\nu}(\mathcal{G}),\ m \in \mathbb{N},\ \nu \in \mathbb{R},\ \nu < 1. \tag{17}$$

For an arbitrary point $x \in \mathcal{G}$, let $i \in \{1, \ldots, n\}$ be such that $x \in (a_i, b_i)$. Then

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m (T_K u)(x) = \sum_{\substack{j=1\\ j\neq i}}^n \int_{a_j}^{b_j} \left(\frac{\partial}{\partial x}\right)^m K(x,y)u(y)\mathrm{d}y + \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \int_{a_i}^{b_i} K(x,y)u(y)\mathrm{d}y, \quad a_i < x < b_i,\ i \in \{1, \ldots, n\}. \tag{18}$$

Introduce a cutting function τ such that

$$\begin{aligned} &\tau \in C^m[0,\infty), \quad 0 \le \tau(r) \le 1 \text{ for } r \ge 0, \\ &\tau(r) = 0 \quad \text{for } 0 \le r \le \frac{1}{2}, \qquad \tau(r) = 1 \quad \text{for } r \ge 1. \end{aligned} \tag{19}$$

Fix an arbitrary point $x' \in (a_i, b_i)$ and denote

$$r' = \frac{1}{2}\rho_{a_i,b_i}(x').$$

For $x \in (a_i, b_i)$, $|x - x'| \le \frac{1}{2}r'$, we represent

$$\int_{a_i}^{b_i} K(x,y)u(y)\mathrm{d}y = \int_{a_i}^{b_i} \tau\left(\frac{|x-y|}{r'}\right) K(x,y)u(y)\mathrm{d}y + \int_{a_i}^{b_i} \left\{1 - \tau\left(\frac{|x-y|}{r'}\right)\right\} K(x,y)u(y)\mathrm{d}y.$$

In the first integral in the right-hand side of the last equality the singularity of $K(x, y)$ at $x = y$ is cut off, therefore we may apply $(\partial/\partial x)$ under the integral sign. In the second integral the coefficient function $1 - \tau(|x - y|/r')$ vanishes for $|x - y| \ge r'$, in particular, for y satisfying $|y - x'| \ge \frac{3}{2}r'$ (since $|x - x'| \le \frac{1}{2}r'$); the boundary points a_i and b_i with their $\frac{1}{2}r'$-neighbourhoods belong to the region where $1 - \tau(|x-y|/r')$ vanishes. Thus in the second integral the boundary singularities caused by $K(x, y)$ are cut off. Due to (3) and Lemma 12, the differentiation formula (16) may be applied obtaining

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{a_i}^{b_i} \left\{1 - \tau\left(\frac{|x-y|}{r'}\right)\right\} K(x,y)u(y)\mathrm{d}y = \int_{a_i}^{b_i} \left\{1 - \tau\left(\frac{|x-y|}{r'}\right)\right\} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \{K(x,y)u(y)\}\mathrm{d}y,$$

where $a_i < x < b_i$. Recall that $1 - \tau(|x-y|/r') = 0$ for $y = a_i$ and $y = b_i$, so that the boundary terms of the formula (16) vanish in our case; we also took into account that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\tau(|x - y|/r') = 0$. In its turn, the last integral may be differentiated in the similar manner. By repeated differentiation we obtain

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \int_{a_i}^{b_i} K(x,y)u(y)\mathrm{d}y = &\int_{a_i}^{b_i} \left(\frac{\partial}{\partial x}\right)^m \left\{\tau\left(\frac{|x-y|}{r'}\right) K(x,y)\right\} u(y)\mathrm{d}y \\ &+ \int_{a_i}^{b_i} \left\{1 - \tau\left(\frac{|x-y|}{r'}\right)\right\} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^m \{K(x,y)u(y)\}\,\mathrm{d}y, \end{aligned}$$

where $a_i < x < b_i$, $|x - x'| \le \frac{1}{2}r'$. Differentiating the product of functions under the integral signs by the Leibniz rule, taking the result at the point $x = x'$ but writing again x instead of x', we obtain the formula

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \int_{a_i}^{b_i} K(x,y)u(y)\mathrm{d}y = &\sum_{k=0}^m \binom{m}{k} \int_{a_i}^{b_i} \tau_{ik}(x,y) \left(\frac{\partial}{\partial x}\right)^{m-k} K(x,y)u(y)\mathrm{d}y \\ &+ \sum_{k=0}^m \binom{m}{k} \int_{a_i}^{b_i} \left\{1 - \tau\left(\frac{2|x-y|}{\rho_i(x)}\right)\right\} \left\{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{m-k} K(x,y)\right\} u^{(k)}(y)\mathrm{d}y, \end{aligned} \tag{20}$$

where $a_i < x < b_i$, $\rho_i(x) \equiv \rho_{a_i,b_i}(x) = \min\{x - a_i, b_i - x\}$ and

$$\tau_{ik}(x,y) = \left[\left(\frac{\partial}{\partial x}\right)^k \tau\left(\frac{|x-y|}{r}\right)\right]_{r=\rho_i(x)/2}, \quad k = 0, \ldots, m. \tag{21}$$

Thus we have the following result.

Lemma 13. *For K and u satisfying (17) the derivative $(\mathrm{d}/\mathrm{d}x)^m(T_K u)(x)$ exists for all $x \in (a_i, b_i)$ $(i = 1, \ldots, n)$ and can be represented in the form (18) where $(\mathrm{d}/\mathrm{d}x)^m \int_{a_i}^{b_i} K(x,y)u(y)\mathrm{d}y$ is given by (20) with a cutting function τ satisfying the conditions (19).*

6. Compactness of T_K in $C^{m,\nu}(\mathcal{G})$

In this section we prove Theorem 4. Assume (17) and let $x \in \mathcal{G}$. Then there exists an integer $i \in \{1, \dots, n\}$ so that $x \in (a_i, b_i)$. Let us multiply both sides of (18) by the weight function $w_{m+\nu-1}^{(a_i,b_i)}$. Due to Lemma 13 the result can be written in the form (see (18)–(21))

$$w_{m+\nu-1}^{(a_i,b_i)} D^m \left[(T_K u)|_{(a_i,b_i)}\right] = \sum_{\substack{j=1\\ j\neq i}}^{n} L_{ij}(u|_{(a_j,b_j)}) + \sum_{k=0}^{m} \binom{m}{k} \{T_{ik}(u|_{(a_i,b_i)}) + S_{ik}(w_{k+\nu-1}^{(a_i,b_i)} D^j (u|_{(a_i,b_i)}))\}, \tag{22}$$

where $D = \frac{d}{dx}$ is the differentiation operator and the operators L_{ij}, T_{ik} and S_{ik} are defined by the following formulas:

$$(L_{ij} v_j)(x) = w_{m+\nu-1}^{(a_i,b_i)}(x) \int_{a_j}^{b_j} \left(\frac{\partial}{\partial x}\right)^m K(x,y) v_j(y) dy, \quad x \in (a_i,b_i),\ v_j \in C^{m,\nu}(a_j,b_j),\ j, i = 1, \dots, n,\ j \neq i; \tag{23}$$

$$(T_{ik} v_i)(x) = w_{m+\nu-1}^{(a_i,b_i)}(x) \int_{a_i}^{b_i} \tau_{ik}(x,y) \left(\frac{\partial}{\partial x}\right)^{m-k} K(x,y) v_i(y) dy,$$
$$k = 0, 1, \dots, m,\ v_i \in C^{m,\nu}(a_i,b_i),\ x \in (a_i,b_i),\ i = 1, \dots, n; \tag{24}$$

$$(S_{ik} v_i)(x) = \int_{a_i}^{b_i} \frac{w_{m+\nu-1}^{(a_i,b_i)}(x)}{w_{k+\nu-1}^{(a_i,b_i)}(y)} \left[1 - \tau\left(\frac{2|x-y|}{\rho_i(x)}\right)\right] \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{m-k} K(x,y)\right] v_i(y) dy,$$
$$k = 0, 1, \dots, m,\ v_i \in BC(a_i,b_i),\ \rho_i(x) = \rho_{a_i,b_i}(x),\ x \in (a_i,b_i),\ i = 1, \dots, n. \tag{25}$$

Note that in (22)

$$\sup_{a_i<y<b_i} w_{k+\nu-1}^{(a_i,b_i)}(y) |(u|_{(a_i,b_i)})^{(k)}(y)| \leq \|u|_{(a_i,b_i)}\|_{C^{m,\nu}(a_i,b_i)}, \quad k = 0, 1, \dots, m. \tag{26}$$

Now the proof of the compactness of $T_K : C^{m,\nu}(\mathcal{G}) \to C^{m,\nu}(\mathcal{G})$ can be reduced to the study of mapping properties of L_{ij}, T_{ik} and S_{ik}. Taking into account Lemma 2 and the compactness of the imbedding $C^{m,\nu}(a_i,b_i) \subset C[a_i,b_i]$, $i = 1, \dots, n$, we observe that for the compactness of T_K in $C^{m,\nu}(\mathcal{G})$, it is sufficient to establish that

$$L_{ij} : BC(a_j,b_j) \to BC(a_i,b_i), \quad j, i = 1, \dots, n, j \neq i, \text{ are bounded}, \tag{27}$$

$$T_{ik} : C^{m,\nu}(a_i,b_i) \to BC(a_i,b_i) \quad \text{are compact}, \tag{28}$$

$$S_{ik} : BC(a_i,b_i) \to BC(a_i,b_i) \quad \text{are compact}, \tag{29}$$

with $k = 0, 1, \dots, m$, $i = 1, \dots, n$.

Let $v_j \in BC(a_j,b_j)$, $x \in (a_i,b_i)$, $j, i \in \{1, \dots, n\}$, $j \neq i$. Then (3), (7) and (23) yield

$$\begin{aligned}
|(L_{ij} v_j)(x)| &\leq c w_{m+\nu-1}^{(a_i,b_i)}(x) \int_{a_j}^{b_j} \left\{\begin{array}{ll} 1 & \text{if } m+\nu<0 \\ 1 + |\log|x-y|| & \text{if } m+\nu=0 \\ |x-y|^{-\nu-m} & \text{if } m+\nu>0 \end{array}\right\} dy \|v_j\|_{BC(a_j,b_j)} \\
&\leq c_1 w_{m+\nu-1}^{(a_i,b_i)}(x) \left[\left\{\begin{array}{ll} 1 & \text{if } m+\nu-1<0 \\ 1 + |\log|x-a_j|| & \text{if } m+\nu-1=0 \\ |x-a_j|^{-\nu-m+1} & \text{if } m+\nu-1>0 \end{array}\right\}\right. \\
&\quad \left. + \left\{\begin{array}{ll} 1 & \text{if } m+\nu-1<0 \\ 1 + |\log|x-b_j|| & \text{if } m+\nu-1=0 \\ |x-b_j|^{-\nu-m+1} & \text{if } m+\nu-1>0 \end{array}\right\}\right] \|v_j\|_{BC(a_j,b_j)} \\
&\leq c_2 \|v_j\|_{BC(a_j,b_j)}, \quad a_i < x < b_i.
\end{aligned}$$

This proves (27).

Next we prove (29). In (25), due to (19),

$$1 - \tau\left(\frac{2|x-y|}{\rho_i(x)}\right) = 0 \quad \text{for } |x-y| \geq \frac{\rho_i(x)}{2},\ i = 1, \dots, n.$$

Thus the integration interval in (25) actually is $(x - \frac{\rho_i(x)}{2}, x + \frac{\rho_i(x)}{2}) \subset (a_i,b_i)$. For $y \in (x - \frac{\rho_i(x)}{2}, x + \frac{\rho_i(x)}{2})$, $x \in (a_i,b_i)$, it holds

$$\frac{1}{2}\rho_i(x) \leq \rho_i(y) \leq \frac{3}{2}\rho_i(x)$$

that implies similar inequalities for the weight functions (7):

$$c_1 w_{k+\nu-1}^{(a_i,b_i)}(x) \leq w_{k+\nu-1}^{(a_i,b_i)}(y) \leq c_2 w_{k+\nu-1}^{(a_i,b_i)}(x), \quad k = 0, \dots, m, i = 1, \dots, n, \tag{30}$$

with some constants $c_2 \geq c_1 > 0$ that are independent of $x \in (a_i, b_i)$ and $y \in (x - \frac{\rho_i(x)}{2}, x + \frac{\rho_i(x)}{2})$. Thus, the functions

$$\frac{w_{m+\nu-1}^{(a_i,b_i)}(x)}{w_{k+\nu-1}^{(a_i,b_i)}(y)}\left[1-\tau\left(\frac{2|x-y|}{\rho_i(x)}\right)\right], \quad k=0,1,\ldots,m,$$

are continuous and bounded for $(x, y) \in (a_i, b_i) \times (a_i, b_i)$, $i = 1, \ldots, n$. Further, due to (3),

$$\left|\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{m-k} K(x,y)\right| \leq c \begin{cases} 1 & \text{if } \nu < 0, \\ 1+|\log|x-y|| & \text{if } \nu = 0, \\ |x-y|^{-\nu} & \text{if } \nu > 0, \end{cases}$$

where $(x, y) \in ((a_i, b_i) \times (a_i, b_i)) \setminus \text{diag}$, $i = 1, \ldots, n$, $k = 0, 1, \ldots, m$. These observations yield (29) since the kernel

$$\frac{w_{m+\nu-1}^{(a_i,b_i)}(x)}{w_{k+\nu-1}^{(a_i,b_i)}(y)}\left[1-\tau\left(\frac{2|x-y|}{\rho_i(x)}\right)\right]\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{m-k} K(x,y)$$

of the integral operator S_{ik} is at most weakly singular for $x = y$.

It remains to prove (28). In the following, we consider an arbitrary i, $1 \leq i \leq n$, i.e. we assume that

$$i \in \{1, \ldots, n\} \quad \text{is fixed.}$$

We obtain from (19) and (21) that,

$$\tau_{ik}(x,y) = 0 \quad \text{if } x \in (a_i,b_i), \qquad |x-y| \leq \frac{\rho_i(x)}{4}, \quad k=0,1,\ldots,m, \tag{31}$$

$$\tau_{ik}(x,y) = 0 \quad \text{if } x \in (a_i,b_i), \qquad |x-y| \geq \frac{\rho_i(x)}{2}, \quad k=1,\ldots,m. \tag{32}$$

Hence

$$\operatorname{supp} \tau_{i0}(x,y) \subset \left\{(x,y) \in [a_i,b_i]\times[a_i,b_i] : |x-y| \geq \frac{\rho_i(x)}{4}\right\}, \tag{33}$$

$$\operatorname{supp} \tau_{ik} \subset \left\{(x,y) \in [a_i,b_i]\times[a_i,b_i] : \frac{\rho_i(x)}{4} \leq |x-y| \leq \frac{\rho_i(x)}{2}\right\}, \quad k=1,\ldots,m, \tag{34}$$

$$|\tau_{ik}(x,y)| \leq c|x-y|^{-k}, \ (x,y) \in \operatorname{supp} \tau_{ik}, \quad k=0,1,\ldots,m. \tag{35}$$

Let $0 < \nu < 1$. Then (3), (7) and (31)–(35) yield

$$\int_{a_i}^{b_i} w_{m+\nu-1}^{(a_i,b_i)}(x)|\tau_{ik}(x,y)|\left|\left(\frac{\partial}{\partial x}\right)^{m-k} K(x,y)\right| dy$$
$$\leq c\,[\rho_i(x)]^{m+\nu-1}\int_{|x-y|\geq\frac{\rho_i(x)}{4}} |x-y|^{-\nu-m}dy \leq c', \quad a_i < x < b_i, k=0,1,\ldots,m.$$

This means that T_{ik} is bounded as an operator from $BC(a_i, b_i)$ into $BC(a_i, b_i)$ for any $k = 0, 1, \ldots, m$. Now the compact imbedding $C^{m,\nu}(a_i, b_i) \subset C[a_i, b_i]$ yields (28) for $0 < \nu < 1$.

Let $\nu = 0$. Then in a similar way as in the case $0 < \nu < 1$, we obtain that T_{ik} is compact as an operator from $C^{m,0}(a_i, b_i)$ into $BC(a_i, b_i)$ for any $k = 0, 1, \ldots, m - 1$. If $k = m$, then $T_{im} : BC(a_i, b_i) \to BC(a_i, b_i)$ is still bounded (and hence $T_{im} : C^{m,0}(a_i, b_i) \to BC(a_i, b_i)$ is compact) for $m = 1$ but not for $m > 1$. Indeed, let $\nu = 0$, $k = m$. Then it follows from (3), (7), (34) and (35) that

$$\int_{a_i}^{b_i} w_{m-1}^{(a_i,b_i)}(x)|\tau_{im}(x,y)||K(x,y)|dy \leq c w_{m-1}^{(a_i,b_i)}(x)\int_{\frac{\rho_i(x)}{4}\leq|x-y|\leq\frac{\rho_i(x)}{2}} |x-y|^{-m}(1+|\log|x-y||)dy$$
$$\leq c' \begin{Bmatrix} 1 & \text{if } m=1 \\ 1+|\log\rho_i(x)| & \text{if } m>1 \end{Bmatrix}, \quad a_i < x < b_i.$$

To prove the compactness of $T_{im} : C^{m,0}(a_i, b_i) \to BC(a_i, b_i)$, $m \geq 2$, we need to treat (24) by using integration by parts.

Let $v_i \in C^{m,0}(a_i, b_i)$, $m \geq 2$. Using (19) and (21) we obtain

$$\tau_{im}(x,y) = -\frac{\partial}{\partial y}\tau_{i,m-1}(x,y), \quad x,y \in (a_i,b_i),$$

$$\tau_{i,m-1}(x,a_i) = \tau_{i,m-1}(x,b_i) = 0, \quad a_i < x < b_i.$$

These observations with integration by parts yield

$$(T_{im}v_i)(x) = \int_{a_i}^{b_i} w_{m-1}^{(a_i,b_i)}(x)\left[-\frac{\partial}{\partial y}\tau_{i,m-1}(x,y)\right]K(x,y)v_i(y)dy$$
$$= (T'_{im}v_i)(x) + (T''_{im}v_i)(x), \quad a_i < x < b_i, \tag{36}$$

where

$$(T'_{im}v_i)(x) = \int_{a_i}^{b_i} w_{m-1}^{(a_i,b_i)}(x)\tau_{i,m-1}(x,y)\left[\frac{\partial}{\partial y}K(x,y)\right]v_i(y)\mathrm{d}y,$$

$$(T''_{im}v_i)(x) = \int_{a_i}^{b_i} w_{m-1}^{(a_i,b_i)}(x)\tau_{i,m-1}(x,y)K(x,y)v'_i(y)\mathrm{d}y. \tag{37}$$

Due to (4), (7), (34) and (35),

$$\int_{a_i}^{b_i} w_{m-1}^{(a_i,b_i)}(x)|\tau_{i,m-1}(x,y)|\left|\frac{\partial}{\partial y}K(x,y)\right|\mathrm{d}y \le c[\rho_i(x)]^{m-1}\int_{\frac{\rho_i(x)}{4}\le|x-y|\le\frac{\rho_i(x)}{2}}|x-y|^{-m}\mathrm{d}y \le c', \quad a_i < x < b_i.$$

Thus, T'_{im} is bounded as an operator from $BC(a_i,b_i)$ into $BC(a_i,b_i)$ for any $m \ge 2$. Since the imbedding $C^{m,0}(a_i,b_i) \subset C[a_i,b_i]$ is compact,

$$T'_{im} : C^{m,0}(a_i,b_i) \to BC(a_i,b_i) \quad \text{is compact for } m \ge 2. \tag{38}$$

To prove the compactness of $T''_{im} : C^{m,0}(a_i,b_i) \to BC(a_i,b_i)$, we observe that

$$\sup_{a_i<y<b_i} w_0^{(a_i,b_i)}(y)|v'_i(y)| \le \|v_i\|_{C^{m,\nu}(a_i,b_i)}. \tag{39}$$

Introduce also an integral operator T'''_{im}, setting

$$(T'''_{im}z_i)(x) = \int_{a_i}^{b_i} H_{im}(x,y)z_i(y)\mathrm{d}y, \quad z_i \in BC(a_i,b_i), \tag{40}$$

$$H_{im}(x,y) = \frac{w_{m-1}^{(a_i,b_i)}(x)}{w_0^{(a_i,b_i)}(y)}\tau_{i,m-1}(x,y)K(x,y), \quad x,y \in (a_i,b_i),\ m \ge 2.$$

Taking into account (3), (7), (30), (34) and (35), we see that the kernel $H_{im}(x,y)$ is at most weakly singular at $x = y$:

$$\begin{aligned}|H_{im}(x,y)| &\le c[\rho_i(x)]^{m-1}(1+|\log\rho_i(x)|)[\rho_i(x)]^{-m+1}(1+|\log\rho_i(x)|)\\ &\le c'(1+|\log|x-y||)^2, \quad (x,y) \in \operatorname{supp}\tau_{i,m-1},\ m \ge 2.\end{aligned}$$

Therefore $T'''_{im} : BC(a_i,b_i) \to BC(a_i,b_i)$ is compact for $m \ge 2$. This together with (37) and (39) yields that T''_{im} is compact as an operator from $C^{m,0}(a_i,b_i)$ into $BC(a_i,b_i)$, $m \ge 2$. Now the compactness of $T_{im} : C^{m,0}(a_i,b_i) \to BC(a_i,b_i)$ $(m \ge 2)$ follows from (36) and (38).

Thus, we have proven (28) (and hence also Theorem 4) for $0 \le \nu < 1$.

For $\nu < 0$ the statement (28) can be established in a similar way (using integration by parts in (24) a suitable number of times). We do not go into the details here. Instead we demonstrate another idea, how Theorem 4 can be proven for $\nu < 0$ using the established results for $0 \le \nu < 1$.

Let $-1 \le \nu < 0$ and let $u_i = u|_{(a_i,b_i)}$ be the restriction of $u \in C^{m,\nu}(\mathcal{G})$ to the interval (a_i,b_i). Then u_i can be extended so that this extension (which we denote again by u_i) is continuously differentiable on $[a_i,b_i]$ and Lemma 12 yields (see Sections 3 and 5 and Remark 1)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{a_i}^{b_i} K(x,y)u_i(y)\mathrm{d}y &= \int_{a_i}^{b_i}\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)K(x,y)\right]u_i(y)\mathrm{d}y\\ &\quad + \int_{a_i}^{b_i} K(x,y)u'_i(y)\mathrm{d}y + u_i(a_i)K(x,a_i) - u_i(b_i)K(x,b_i), \quad a_i < x < b_i,\end{aligned}$$

or

$$DT_iu_i = T_i^{(1)}u_i + T_iDu_i + R_iu_i,$$

where

$$(T_iu_i)(x) = \int_{a_i}^{b_i} K(x,y)\,u_i(y)\mathrm{d}y, \ \ a_i < x < b_i,$$

$$(T_i^{(1)}u_i)(x) = \int_{a_i}^{b_i}\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)K(x,y)\right]u_i(y)\mathrm{d}y, \quad a_i < x < b_i,$$

$$(R_iu_i)(x) = u_i(a_i)K(x,a_i) - u_i(b_i)K(x,b_i), \quad a_i < x < b_i. \tag{41}$$

This together with (18) and (23) yields

$$w_{m+\nu-1}^{(a_i,b_i)}D^mT_iu_i = \sum_{j=1,j\ne i}^{n} L_{ij}u_j + w_{m+\nu-1}^{(a_i,b_i)}(D^{m-1}T_iDu_i + D^{m-1}T_i^{(1)}u_i + D^{m-1}R_iu_i). \tag{42}$$

Since the imbedding $C^{m,\nu}(a_i, b_i) \subset C[a_i, b_i]$ is compact, it follows from (27) that L_{ij} is compact as an operator from $C^{m,\nu}(a_i, b_i)$ into $BC(a_i, b_i)$ for $j = 1, \dots, n, j \neq i$. Further, if $u_i \in C^{m,\nu}(a_i, b_i)$, $-1 \le \nu < 0$, then $Du_i \in C^{m-1,\nu+1}(a_i, b_i)$ with $\nu + 1 \in [0, 1)$ and

$$\|Du_i\|_{C^{m-1,\nu+1}(a_i,b_i)} \le \|u_i\|_{C^{m,\nu}(a_i,b_i)}, \quad -1 \le \nu < 0.$$

Due to results obtained above for the case $\nu + 1 \in [0, 1)$, the operator

$$w^{(a_i,b_i)}_{m+\nu-1} D^{m-1} T_i = w^{(a_i,b_i)}_{(m-1)+(\nu+1)-1} D^{m-1} T_i$$

is compact as an operator from $C^{m-1,\nu+1}(a_i, b_i)$ into $BC(a_i, b_i)$, $-1 \le \nu < 0$. Hence $w^{(a_i,b_i)}_{m+\nu-1} D^{m-1} TD$ is compact as an operator from $C^{m,\nu}(a_i, b_i)$ into $BC(a_i, b_i)$, $-1 \le \nu < 0$.

Further, it follows from (3) that the kernel $(\partial/\partial x + \partial/\partial y)K(x, y)$ of the operator $T^{(1)}_i$ satisfies same inequalities as the kernel $K(x, y)$ of the operator T_i. Therefore $w^{(a_i,b_i)}_{m+\nu-1} D^{m-1} T^{(1)}_i$ is also compact as an operator from $C^{m,\nu}(a_i, b_i)$ into $BC(a_i, b_i)$, $-1 \le \nu < 0$.

Finally, the compactness of

$$w^{(a_i,b_i)}_{m+\nu-1} D^{m-1} R_i : C^{m,\nu}(a_i, b_i) \to BC(a_i, b_i)$$

for $-1 \le \nu < 0$ is a consequence of the boundedness of this finite-dimensional operator (note that R_i is a two-dimensional operator for any $i = 1, \dots, n$, see (41)). These observations together with (42) and Lemma 2 yield that

$$w^{(a_i,b_i)}_{m+\nu-1} D^{m} T_i : C^{m,\nu}(a_i, b_i) \to BC(a_i, b_i) \quad \text{is compact,}$$

for $-1 \le \nu < 0$. In a similar way as above for the case $0 \le \nu < 1$, we obtain from this the claim of Theorem 4 for $-1 \le \nu < 0$.

Having proved Theorem 4 for $\nu \in [-1, 0)$, in a similar way we extend the claim for $\nu \in [-2, -1)$ etc. Theorem 4 is proven. □

7. Compactness of T_K in $C^{m,\nu,p}(\mathcal{G}_d)$

In this Section we prove Theorem 9. Let $\mathcal{G} = \mathcal{G}_d$, $\mathcal{G}_d = (a, b) \setminus \{d\}$, $a < d < b$, $K \in \mathcal{W}^{m,\nu}((\mathcal{G}_d \times \mathcal{G}_d) \setminus \text{diag}) \cap C^{p-1}(((a, b) \times (a, b)) \setminus \text{diag})$, $u \in C^{m,\nu,p}(\mathcal{G}_d)$, $m, p \in \mathbb{N}$, $p \le m$, $\nu \in \mathbb{R}$, $\nu < 1$. Let T_K be defined by (11). Our purpose is to show that

$$T_K u \in C^{m,\nu,p}(\mathcal{G}_d), \tag{43}$$

$$T_K : C^{m,\nu,p}(\mathcal{G}_d) \to C^{m,\nu,p}(\mathcal{G}_d) \quad \text{is compact.} \tag{44}$$

Let a_i, b_i $(i = 1, \dots, 4)$ be some real numbers such that

$$a < a_1 < a_2 < a_3 < a_4 < d, \qquad d < b_4 < b_3 < b_2 < b_1 < b.$$

Introduce two cutting functions $\sigma, \tilde{\sigma} \in C^m[a, b]$ such that

$$\begin{aligned}
&0 \le \sigma(x) \le 1 \qquad 0 \le \tilde{\sigma}(x) \le 1, \quad \text{for } a \le x \le b,\\
&\sigma(x) = 0 \quad \text{for } a_4 \le x \le b_4,\\
&\sigma(x) = 1 \quad \text{for } a \le x \le a_3,\ b_3 \le x \le b,\\
&\tilde{\sigma}(x) = 0 \quad \text{for } a_2 \le x \le b_2,\\
&\tilde{\sigma}(x) = 1 \quad \text{for } a \le x \le a_1,\ b_1 \le x \le b.
\end{aligned} \tag{45}$$

It follows from this that for $x, y \in [a, b]$,

$$[1 - \sigma(x)]\, \tilde{\sigma}(y) = 0 \quad \text{if } |x - y| \le \min\{a_3 - a_2, b_2 - b_3\}. \tag{46}$$

Using σ and $\tilde{\sigma}$ we represent $T_K u$ in the form

$$T_K u = \sigma T_K u + (1 - \sigma) T_K \tilde{\sigma} u + (1 - \sigma) T_K (1 - \tilde{\sigma}) u. \tag{47}$$

Since $K \in \mathcal{W}^{m,\nu}((\mathcal{G}_d \times \mathcal{G}_d) \setminus \text{diag})$, it follows from (13) and Theorem 4 that $T_K u \in C^{m,\nu}(\mathcal{G}_d)$ and $T_K : C^{m,\nu}(\mathcal{G}_d) \to C^{m,\nu}(\mathcal{G}_d)$ is compact. Thus,

$$\sigma T_K u \in C^{m,\nu,p}(\mathcal{G}_d), \tag{48}$$

$$\sigma T_K : C^{m,\nu,p}(\mathcal{G}_d) \to C^{m,\nu,p}(\mathcal{G}_d) \quad \text{is compact.} \tag{49}$$

Further, we have

$$[(1 - \sigma) T_K \tilde{\sigma} u](x) = \int_a^b [1 - \sigma(x)] K(x, y) \tilde{\sigma}(y) u(y) \mathrm{d}y, \quad x \in \mathcal{G}_d.$$

On the basis of (46) we obtain that the function $[1 - \sigma(x)]K(x, y)\tilde{\sigma}(y)$ is m times continuously differentiable for $(x, y) \in [a, b] \times [a, b]$. This yields that $(1 - \sigma) T_K \tilde{\sigma} u \in C^m[a, b]$ and $(1 - \sigma) T_K \tilde{\sigma}$ is compact as an operator from $L^\infty(a, b)$ into $C^m[a, b] \subset C^{m,\nu,p}(\mathcal{G}_d)$. Thus, we get

$$(1 - \sigma) T_K \tilde{\sigma} u \in C^{m,\nu,p}(\mathcal{G}_d), \tag{50}$$

$$(1 - \sigma) T_K \tilde{\sigma} : C^{m,\nu,p}(\mathcal{G}_d) \to C^{m,\nu,p}(\mathcal{G}_d) \quad \text{is compact.} \tag{51}$$

Let us consider the last term in the expansion (47):

$$[(1-\sigma)T_K(1-\tilde{\sigma})u](x) = \int_a^b [1-\sigma(x)]K(x,y)[1-\tilde{\sigma}(y)]u(y)\mathrm{d}y, \quad x \in \mathcal{G}_d.$$

It follows from (45) that

$$[1-\sigma(x)]K(x,y)[1-\tilde{\sigma}(y)] = 0$$

for $(x,y) \in ([a,b]\times[a,b]) \setminus ((a_1,b_1)\times(a_1,b_1))$. Moreover, we observe that

$$(1-\sigma)K(1-\tilde{\sigma}) \in \mathcal{W}^{m,\nu}((\mathcal{G}_d \times \mathcal{G}_d)\setminus \mathrm{diag}) \cap C^{p-1}(([a,b]\times[a,b])\setminus \mathrm{diag}).$$

Therefore, due to (3) and Lemma 12,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \int_a^b (1-\sigma(x))K(x,y)(1-\tilde{\sigma}(y))u(y)\mathrm{d}y = \int_a^b \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^k [(1-\sigma(x))K(x,y)(1-\tilde{\sigma}(y))u(y)]\,\mathrm{d}y$$
$$= \sum_{j=0}^{k}\binom{k}{j}\int_a^b \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{k-j}[(1-\sigma(x))K(x,y)]\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^j[(1-\tilde{\sigma}(y))u(y)]\,\mathrm{d}y,$$

where $x \in \mathcal{G}_d$ and $k = 0,1,\ldots,p$. This together with (3) and $(1-\tilde{\sigma})u \in C^p[a,b]$ yields that $(1-\sigma)T_K(1-\tilde{\sigma})u \in C^p[a,b]$. We observe also that $(1-\tilde{\sigma})u \in C^{m,\nu-p}(\mathcal{G}_d)$. Therefore, by Theorem 4 (with $\nu-p$ in the role of ν),

$$T_K(1-\tilde{\sigma})u \in C^{m,\nu-p}(\mathcal{G}_d)$$

and T_K is compact as an operator from $C^{m,\nu-p}(\mathcal{G}_d)$ into $C^{m,\nu-p}(\mathcal{G}_d)$. Thus,

$$(1-\sigma)T_K(1-\tilde{\sigma})u \in C^{m,\nu-p}(\mathcal{G}_d)\cap C^p[a,b] \subset C^{m,\nu,p}(\mathcal{G}_d)$$

and $(1-\sigma)T_K(1-\tilde{\sigma})$ is compact as an operator from $C^{m,\nu,p}(\mathcal{G}_d)$ into $C^{m,\nu,p}(\mathcal{G}_d)$. This together with (47)–(51) yields (43) and (44).

Theorem 9 is proven. □

8. The proof of Lemma 6

The proof of Lemma 6 is given in [28]. Since [28] is not easy to find, here we shortly repeat the argument of [28].

Let E and F be Banach spaces such that $E \subset F$ densely and continuously. Then for the dual spaces E' and F', $F' \subset E'$ holds continuously. Further, let $T : F \to F$ be a linear compact operator such that T maps E into E and the restriction $T_E : E \to E$ is compact as an operator in E. Then the dual operator $T' : F' \to F'$ is the restriction of $T'_E : E' \to E'$ to the space F'. Denote

$$\begin{aligned} N(I-T) &= \{v \in F : v - Tv = 0\} \subset F,\\ N(I-T_E) &= \{u \in E : u - T_E u = 0\} \subset E,\\ N(I-T') &= \{v' \in F' : v' - T'v' = 0\} \subset F',\\ N(I-T'_E) &= \{u' \in E' : u' - T_E u' = 0\} \subset E'. \end{aligned} \tag{52}$$

Clearly,

$$N(I-T_E) \subset N(I-T), \qquad N(I-T') \subset N(I-T'_E). \tag{53}$$

Moreover, due to the compactness of T and T_E, the subspaces (52) are finite dimensional and

$$\begin{aligned} \dim\, N(I-T) &= \dim\, N(I-T'),\\ \dim\, N(I-T_E) &= \dim\, N(I-T'_E). \end{aligned} \tag{54}$$

It follows from (53) and (54) that

$$\begin{aligned} \dim\, N(I-T_E) &\le \dim\, N(I-T) = \dim\, N(I-T')\\ &\le \dim\, N(I-T'_E) = \dim\, N(I-T_E). \end{aligned}$$

Hence

$$\dim\, N(I-T_E) = \dim\, N(I-T), \qquad \dim\, N(I-T') = \dim\, N(I-T'_E).$$

This together with (53) yields

$$N(I-T_E) = N(I-T), \qquad N(I-T') = N(I-T'_E). \tag{55}$$

By the hypothesis of Lemma 6, the equation $v = Tv + f$, with a given $f \in E \subset F$, has a solution $\tilde{v} \in F$. Therefore

$$\langle f, v'\rangle = 0 \quad \text{for every } v' \in N(I-T').$$

On the basis of (55) we now obtain that

$$\langle f, u' \rangle = 0 \quad \text{for every } u' \in N(I - T'_E).$$

This implies the solvability of equation $u = T_E u + f$ in E.

Let $\tilde{u} \in E \subset F$ be a solution of equation $u = T_E u + f$. Then $\tilde{u}$ is a solution of equation $v = Tv + f$, also. Now we have

$$\tilde{v} - \tilde{u} \in N(I - T) = N(I - T_E) \subset E.$$

This together with $\tilde{u} \in E$ yields $\tilde{v} \in E$.

The Lemma is proven. □

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